Why is Landau damping important?

- A priori « collisionless plasma » ==> no damping of Langmuir waves
- Landau 1946: « They are damped ! »
- Plasma community: « Why ? This is unbelievable ! »
- Malmberg and Wharton 1964: « Look, they are damped experimentally ! »
- Plasma community: « OK, we accept, but why ? This is still unbelievable ! »
- In textbooks many little small mechanical models... which disagree, and several are wrong!
Why is Landau damping important?

\[ \gamma_L(k_m) = \frac{\pi \omega_P^2}{k_m^2 \frac{\partial \varepsilon}{\partial \omega}} g'(\frac{\omega_{mr}}{k_m}) \]

Energetic geodesic acoustic modes in tokamaks Zarzoso 2012

Landau damping/instability also in many media with a continuous oscillation spectrum (Vekstein 1998):
- at the root of wind-generated water waves
- exists in liquids with gas bubbles, in high-energy particle beams, in superfluids, and in quark–gluon plasmas

Analogs of Landau damping also in biology in connection with:
- the flashing of fireflies
- the pacemaker cells controlling the beating of the heart
Pedestrian derivation of Landau damping/growth by $N$-body mechanics (à la Kaufman 1972)

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One calculation focussing on waves
One calculation focussing on particles: exhibits their average synchronization with the wave
One Component Plasma model

- Infinite plasma with spatial periodicity $L$ in three orthogonal directions
- Made up of $N$ electrons in each elementary cube with volume $L^3$
- Uniform ionic neutralizing background
\[ n = 10^{19} \text{ m}^{-3} \quad T = 1 \text{ keV} \]

\[ \lambda_{ca} = \frac{e^2}{4\pi \varepsilon_0 T} \quad \lambda_{id} = n^{-1/3} \quad \lambda_D = \frac{\nu_T}{\omega_p} = \left[ \frac{\varepsilon_0 T}{ne^2} \right]^{1/2} \]
Perturbation theory similar to the Vlasovian one

- Almost uniform distributions of particles: set of monokinetic beams; each beam is a simple cubic array of particles

Wikipedia: Perturbation theory comprises mathematical methods for finding an approximate solution to a problem, by starting from the exact solution of a related, simpler problem

Here: many-beam solution + first-order perturbation correction for the N-body motion

\[ r = r_0 + v_0 t + \Delta r \]
“Spatially uniform” granular distribution of particles

- Multi-beam-multi-array

\[ \tilde{\varphi}(m) = -\frac{e}{\varepsilon_0 k_m^2} \sum_{j=1}^{N} \exp[-ik_m \cdot r_j(t)] \]

- All \( \tilde{\varphi}(m) \)‘s vanish

- So does \( \varphi(r) = \frac{1}{L^3} \sum_{m, k_m b_{smooth} \leq 1} \tilde{\varphi}(m) \exp(ik_m \cdot r) \)

and the acceleration of particles

- Each array propagates without any perturbation: collisions do not work!

- This distribution is analogous to \( f_0(\nu) \) in Vlasovian calculations

- ==> we can develop an analogous perturbation theory
Three derivations of Landau damping by N-body mechanics

• A pedestrian, short, yet rigorous, one germane to Kaufman’s in a Vlasovian setting (1972)

Unambiguous mechanical understanding of Landau damping & growth: average synchronization of particles with the wave

• One showing waves damp because of phase mixing à la van Kampen (1955)

• One showing that the Vlasovian limit is singular and corresponds to a renormalized description of the actual N-body dynamics
Phase mixing

With many smaller arrows, there is a damping when time increases: Destructive interference

Mathematical justification in Fourier transform; see RMPP

Footnote 8 page 12 of RMPP

Arrows rotating with increasing rotation frequency
Essence of the calculation for waves

• Look for a wave with a given $k$ and an amplitude $A(t) \exp(i \omega t)$, with $\omega$ and $A$ unknown
  $A$ slow (evolves over a time-scale $\gg \omega^{-1}$)

• Integrate formally the linearized equations of motion of the particles; $r = r_0 + v_0 t + \Delta r$

• Set the $\Delta r$’s into the perturbed Coulomb potential

• Substitute the discrete sum over particles with the integral over a smooth distribution function

• A rigorous Taylor expansion yields an expression for asymptotic times (after phase mixing) defining both $\omega$ and $dA/dt$ with their Vlasovian values in Landau’s calculation: Classical Vlasovian expression recovered
• Electrostatic potential $A(t) \exp(i \omega t)$  
  \[ A(t) = A_0 \exp(\gamma t) \]

• Classical Vlasovian Result (Landau 1946)

  \[ \varepsilon_r(m, \omega) = 0. \]

  \[ \varepsilon_r(m, \omega) = 1 - \frac{\omega_p^2}{k_m} P \int g'(v) \frac{1}{\Omega} \, dv \]

  \[ \Omega = k_m v - \omega \]

• Yields  
  \[ \omega_{mr} \sim \omega_p \]

  \[ \gamma = \gamma_L(k_m) \]

  \[ \gamma_L(k_m) = \frac{\pi \omega_p^2}{k_m} \frac{k_m^2}{\partial \varepsilon_r / \partial \omega} g'(\frac{\omega_{mr}}{k_m}) \]

  Landau damping/growth rate

  Longitudinal Langmuir waves

\[ f(x) \]

slower particles  

faster particles

\[ v_{ph} \]
One calculation focussing on waves

One calculation focussing on particles: exhibits their average synchronization with the wave
Recovering Landau damping by computing particle acceleration

**Principle**

- The dynamics of particles can be dealt with in a way similar to that for the waves.
- Compute the sum of the \( \vec{f}_{ij} \) for particles whose position is slightly perturbed with respect to monokinetic beams.
- This sum vanishes, since the system is isolated: multiplying this sum by \( m \) corresponds to the time derivative of the total momentum of the system, which is conserved.
0 = -\alpha A(t) k_m \sum_{j=1}^{N} \mathbf{k}_m \cdot \delta \mathbf{r}_j \exp(i \Omega_j t) + \text{c.c.}

\begin{align*}
+ \sum_{j=1}^{N} 4|\alpha|^2 k_m^2 k_m \left[ |A(t)|^2 \frac{\partial}{\partial \Omega_j} \frac{\sin(\Omega_j t)}{\Omega_j} \right. \\
\left. + A(t) A^*(t) \frac{\partial^2}{\partial \Omega_j^2} \frac{\exp(i \Omega_j t) - 1}{2 \Omega_j} \right] + \text{c.c.}
\end{align*}

- **Second term**: contribution of particles nearly-resonant with the wave

- **The derivative in \( \Omega \) of \( \frac{\sin(\Omega t)}{\Omega} \) is**

\[ B = [\Omega t \cos(\Omega t) - \sin(\Omega t)] \Omega^{-2} \]

Synchronization also for a monokinetic beam
\[ \mathcal{N} = -\alpha A(t) k_m \sum_{j=1}^{N} k_m \cdot \delta r_j \exp(i\Omega_j t) + \text{c.c.} \]
\[ + \sum_{j=1}^{N} 4|\alpha|^2 k_m^2 k_m \left[ |A(t)|^2 \frac{\partial}{\partial \Omega_j} \frac{\sin(\Omega_j t)}{\Omega_j} \right. \]
\[ \left. + A(t)A^*(t) \frac{\partial^2}{\partial \Omega_j^2} \frac{\exp(i\Omega_j t) - 1}{2\Omega_j} + \text{c.c.} \right] \]

- Contribution of nearly-resonant particles
- The derivative in $\Omega$ of $\frac{\sin(\Omega t)}{\Omega}$ is

\[ B = [\Omega t \cos(\Omega t) - \sin(\Omega t)]\Omega^{-2} \]

- Landau damping occurs over a time scale $\gamma_L^{-1}(k_m)$
- Synchronization is mostly experienced by the particles whose absolute differential velocity to the wave scales as $|\gamma_L|/k_m$
Can also be written as

\[ 0 = -\alpha A(t)k_m \sum_{j=1}^{N} k_m \cdot \delta r_j \exp(i\Omega_j t) + \text{c.c.} \]

\[ + \sum_{j=1}^{N} 4|\alpha|^2 k_m^2 k_m \left[ |A(t)|^2 \frac{\partial}{\partial \Omega_j} \frac{\sin(\Omega_j t)}{\Omega_j} \right. \]

\[ + A(t) A^*(t) \frac{\partial^2}{\partial \Omega_j^2} \frac{\exp(i\Omega_j t) - 1}{2\Omega_j} \left. + \text{c.c.} \right] \]

Pushing forward the calculation of

No dissipation!

Yields

\[ \frac{dP_{\text{res}}}{dt} + \frac{dP_{\text{wave}}}{dt} = 0 \]

\[ \frac{dP_{\text{res}}}{dt} = -\frac{4\pi \varepsilon_0 w_p^2}{L^3} g'(\frac{\omega}{k_m}) |A(t)|^2 k_m \]

\[ P_{\text{wave}} = \frac{2\varepsilon_0 k_m^2}{L^3} \frac{\partial \varepsilon_r(m, \omega)}{\partial \omega} |A(t)|^2 k_m \]

Can also be written as

\[ \frac{d|A(t)|^2}{dt} = 2\gamma_L(k_m) |A(t)|^2 \]

Landau growth or damping: Exchange of momentum of the wave with nearly resonant particles
On average particles synchronize with Langmuir waves: intuitive picture

Larger deflection
Stronger sticking to X-point


Landau damping and growth result from the average synchronization of untrapped particles

All particles do not synchronize

Particles transferring the most momentum to the wave
at $u \sim |\gamma_L|/k$

Since $|\gamma_L| >> \omega_b$, particles are far from being trapped

Weakly resonant effect rapping are wrong

The popular surfer model is misleading:
- All faster particles do not « push » the wave
- Too suggestive of an effect due to trapping
Experimental set up
Experimental synchronization in a travelling wave tube

Recovering Landau damping by computing particle acceleration

More precisely
Step 1: average synchronization of particles with a wave

• Use again

\[
\Delta r_{j1}(t) = \delta r_j \sin(k_m \cdot r_{j0}) + \alpha k_m \int_0^t \tau A(t-\tau) \exp[i(\Omega_j(t-\tau) + k_m \cdot r_{j0})] d\tau + \text{c.c.}
\]

To express \( r_j(t) \) to first order in \( \Delta r_j(t) = r_j(t) - r_{j0} - v_j t \)

\[
\ddot{r}_j = \alpha k_m A(t) \exp[i(k_m \cdot r_j(t) - \omega t)] + \text{c.c.}
\]

• Yields

\[
0 = -\alpha A(t) k_m \sum_{j=1}^N k_m \cdot \delta r_j \exp(i\Omega_j t) + \text{c.c.}
\]

\[
+ \sum_{j=1}^N 4|\alpha|^2 k_m^2 k_m \left[ |A(t)|^2 \frac{\partial}{\partial \Omega_j} \frac{\sin(\Omega_j t)}{\Omega_j} + A(t)A^*(t) \frac{\partial^2}{\partial \Omega_j^2} \frac{\exp(i\Omega_j t) - 1}{2\Omega_j} + \text{c.c.} \right]
\]
\[ 0 = -\alpha A(t) k_m \sum_{j=1}^{N} k_m \cdot \delta r_j \exp(i\Omega_j t) + \text{c.c.} \]

\[ + \sum_{j=1}^{N} 4|\alpha|^2 k_m^2 k_m \left[ |A(t)|^2 \frac{\partial}{\partial \Omega_j} \frac{\sin(\Omega_j t)}{\Omega_j} \right. \]

\[ + A(t)A^*(t) \frac{\partial^2}{\partial \Omega_j^2} \frac{\exp(i\Omega_j t) - 1}{2\Omega_j} + \text{c.c.} \]

- Second term: contribution of particles nearly-resonant with the wave
- Third term contribution of non-resonant particles
Focus on second term
\[ a = \sum_{j=1}^{N} 4|\alpha|^2 k_m^2 k_m \left[ |A(t)|^2 \frac{\partial}{\partial \Omega_j} \frac{\sin(\Omega_j t)}{\Omega_j} \right] \]

corresponding to the contribution of particles close to being resonant with the wave

The derivative in \( \Omega \) of \( \frac{\sin(\Omega t)}{\Omega} \) is
\[ B = [\Omega t \cos(\Omega t) - \sin(\Omega t)] \Omega^{-2}. \]

Therefore, the component of \( a \) along \( k_m \) has the sign of \( B \)

For \( |\Omega|t \ll 1, B \sim -\Omega t^3/3 \)
\[ B = [\Omega t \cos(\Omega t) - \sin(\Omega t)] \Omega^{-2}, \quad |\Omega|t \ll 1, \quad B \approx -\Omega t^3/3 \]
\[ \Omega = k_m v - \omega \]

- Average synchronization: \( a \sim B \sim -\Omega \)
- Stronger when \( |\Omega| \) grows at fixed \( t \)
- The synchronization vanishes with \( |\Omega| \)

Rules out any role of trapped particles in Landau damping or growth

A possible role of trapping is a priori excluded since the bounce period is unbounded in the linear regime of Langmuir waves
\[ B = [\Omega t \cos(\Omega t) - \sin(\Omega t)]\Omega^{-2}, \quad |\Omega|t \ll 1, \ B \approx -|\Omega|t^3/3 \]

- \( \Omega t \) not small: \( B \) keeps the sign of \( -\Omega \) for \( \Omega t \) of the order of a few units

Landau damping occurs over a time scale \( \gamma_L^{-1}(k_m) \)

\[ \Rightarrow \text{synchronization occurs for } |\Omega| \text{ up to the order of } |\gamma_L| \]
\[ B = [\Omega t \cos(\Omega t) - \sin(\Omega t)]\Omega^{-2}, \quad |\Omega| t \ll 1, B \simeq -\Omega t^3/3 \]

- \( \Omega t \) not small: \( B \) keeps the sign of \(-\Omega\) for \( \Omega t \) of the order of a few units

Landau damping occurs over a time scale \( \gamma_L^{-1}(k_m) \)

\[ \Rightarrow \text{synchronization occurs for } |\Omega| \text{ up to the order of } |\gamma_L| \]

- For \( |\Omega| \) large
- \( |B| \) maximum for \( t/\Omega \approx 0.5 \)

Synchronization maximum for \( |\Omega| \) of the order of \( |\gamma_L| \) \( \Omega = k_m v - \omega \)
• Synchronization is mostly experienced by the particles whose absolute differential velocity to the wave scales as $|\gamma_L|/k_m$

• When $|\gamma_L|/k_m$ is small, these particles are quasi-resonant
Step 2: Landau damping & growth

\[ 0 = -\alpha A(t) k_m \sum_{j=1}^{N} k_m \cdot \delta r_j \exp(i\Omega_j t) + \text{c.c.} \]

\[ + \sum_{j=1}^{N} 4|\alpha|^2 k_m^2 k_m \left[ |A(t)|^2 \frac{\partial}{\partial \Omega_j} \frac{\sin(\Omega_j t)}{\Omega_j} \right. \]

\[ + A(t) \dot{A}^*(t) \frac{\partial^2}{\partial \Omega_j^2} \frac{\exp(i\Omega_j t) - 1}{2\Omega_j} + \text{c.c.} \]
\[ 0 = -\alpha A(t) k_m \sum_{j=1}^{N} k_m \cdot \delta r_j \exp(i\Omega_j t) + \text{c.c.} \]
\[ + \sum_{j=1}^{N} 4|\alpha|^2 k_m^2 k_m \left[ |A(t)|^2 \frac{\partial}{\partial \Omega_j} \frac{\sin(\Omega_j t)}{\Omega_j} \right. \]
\[ + A(t) \dot{A}^*(t) \frac{\partial^2}{\partial \Omega_j^2} \frac{\exp(i\Omega_j t) - 1}{2\Omega_j} + \text{c.c.} \left. \right] \]

- Introduce again a smooth distribution function
- With phase mixing for \( t \) large, yields

\[ 0 = -4\pi N |\alpha|^2 k_m g'\left( \frac{\omega}{k_m} \right) |A(t)|^2 + 2N |\alpha|^2 k_m \frac{k_m^2}{\omega_p^2} \frac{\partial \epsilon_r}{\partial \omega} \frac{d |A(t)|^2}{dt} \]

- First term: contribution of particles nearly-resonant with the wave
- Second term contribution of non-resonant particles
- Multiplying by the electron mass \( m \) yields
0 = -4\pi N|\alpha|^2 k_m g'(\frac{\omega}{k_m})|A(t)|^2 + 2N|\alpha|^2 k_m \frac{k_m^2}{\omega_p^2} \frac{\partial \epsilon_\tau}{\partial \omega} \frac{d|A(t)|^2}{dt}

- Multiplying by the electron mass \( m \) yields

\[
\frac{dP_{\text{res}}}{dt} + \frac{dP_{\text{wave}}}{dt} = 0,
\]

Derivative of the momentum of particles nearly resonant with the wave

\[
\frac{dP_{\text{res}}}{dt} = -\frac{4\pi \varepsilon_0 \omega_p^2}{L^3} g'(\frac{\omega}{k_m})|A(t)|^2 k_m
\]

\[
P_{\text{wave}} = \frac{2\varepsilon_0 k_m^2}{L^3} \frac{\partial \epsilon_\tau(m, \omega)}{\partial \omega} |A(t)|^2 k_m
\]

Total wave momentum in the volume \( L^3 \)

- Conservation of the total momentum recovered
\[ \frac{dP_{\text{res}}}{dt} + \frac{dP_{\text{wave}}}{dt} = 0 \quad \frac{dP_{\text{res}}}{dt} = -\frac{4\pi\varepsilon_0\omega_p^2}{L^3} g'(\frac{\omega}{k_m})|A(t)|^2 k_m \]

\[ P_{\text{wave}} = \frac{2\varepsilon_0 k_m^2}{L^3} \frac{\partial \varepsilon_r(m, \omega)}{\partial \omega} |A(t)|^2 k_m \]

can also be written as

\[ \frac{d|A(t)|^2}{dt} = 2\gamma_L(k_m)|A(t)|^2 \]

giving again Landau growth or damping
\[
\frac{dP_{\text{res}}}{dt} + \frac{dP_{\text{wave}}}{dt} = 0
\]
\[
\frac{dP_{\text{res}}}{dt} = -\frac{4\pi \varepsilon_0 \omega_p^2}{L^3} g'(\frac{\omega}{k_m}) |A(t)|^2 k_m
\]
\[
P_{\text{wave}} = \frac{2\varepsilon_0 k_m^2}{L^3} \frac{\partial \epsilon_r(m, \omega)}{\partial \omega} |A(t)|^2 k_m
\]

can also be written as
\[
\gamma_L(k_m) = \frac{\pi \omega_p^2}{k_m^2} \frac{\partial \epsilon_r}{\partial \omega} g'(\frac{\omega_{\text{mr}}}{k_m})
\]
giving again Landau growth or damping.

There is a net loss of particles momentum when
\[
g'(\frac{\omega_{\text{mr}}}{k_m}) > 0
\]
and a net gain of momentum in the opposite case.

Landau effect: a consequence of momentum exchange between the wave and close to resonant particles.
Conclusion

• Strong simplification with respect to the usual textbook calculation

• Calculation of Landau damping accessible to students who only know Newton’s second law and elementary calculus

• The continuous velocity distribution is introduced after particle dynamics has been taken into account, and not before, as occurs in the kinetic approach
• The N-body approach shows rigorously that Landau damping results from the simultaneous average synchronization of almost resonant passing particles with the wave.

Landau damping is not due to trapping.

In contrast with the surfer model, all particles faster than the wave do not «push» the wave, and vice-versa for slow ones.
Experimental synchronization in a travelling wave tube

Doveil, DFE, and Macor

Open issues

• What happens when $b_{\text{smooth}}$ goes to 0?
• What happens if particles have random positions at $t=0$?
• ???
References/1


Landau damping as a phase mixing & Vlasovian description as a singular limit

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Three derivations of Landau damping & growth by N-body mechanics

- A pedestrian, short, yet rigorous, one germane to Kaufman’s in a Vlasovian setting (1972)
- Unambiguous mechanical understanding of Landau damping/growth: average synchronization of particles with the wave
- One showing waves damp because of phase mixing à la van Kampen (1955)
- One showing that the Vlasovian limit is singular and corresponds to a renormalized description of the actual N-body dynamics
- Both need the derivation of a fundamental equation for the electrostatic potential
Essence of the derivation of the fundamental equation for the electrostatic potential (1)

• Perturbation from ballistic orbits of electrons
  \[ r = r_0 + v_0 t + \delta r \]

• Laplace transform the perturbed Coulomb potential

• Laplace transform Newton equation for the particles acted upon by this potential
  \[ \hat{g}(\omega) = \int_0^\infty g(t) \exp(i\omega t) dt \]
  \( \omega \) complex

• Combine these equations

• Use that particles sit on spatial arrays: big cancellations!

• Needs enough beam particles
Essence of the derivation of the fundamental equation for the electrostatic potential (2)

- Yields

\[ k_m = \frac{2\pi}{L} \, m \]

\[ \epsilon_d(m, \omega) = 1 - \frac{\omega_p^2}{N} \sum_{j=1}^{N} \frac{1}{(\omega - k_m \cdot v_j)^2} \]

\[ \varphi^{(bal)}(m, \omega) = \sum_{j=1}^{N} \varphi_j^{(bal)}(m, \omega), \]

\[ \varphi_j^{(bal)}(m, \omega) = -\frac{ie}{\varepsilon_0 k_m^2} \frac{\exp[-ik_m \cdot r_j(0)]}{\omega - k_m \cdot \dot{r}_j(0)} \]
Fundamental equation for the electrostatic potential

\[ \epsilon_d(m, \omega) \varphi(m, \omega) = \varphi^{(bal)}(m, \omega) \]

\[ k_m = \frac{2\pi}{L} m \]

\[ \epsilon_d(m, \omega) = 1 - \frac{\omega_p^2}{N} \sum_{j=1}^{N} \frac{1}{(\omega - k_m \cdot v_j)^2} \]

\[ \varphi^{(bal)}(m, \omega) = \sum_{j=1}^{N} \varphi_j^{(bal)}(m, \omega), \]

\[ \varphi_j^{(bal)}(m, \omega) = -\frac{ie}{\varepsilon_0 k_m^2} \exp[-ik_m \cdot r_j(0)] \frac{\exp[-ik_m \cdot \dot{r}_j(0)]}{\omega - k_m \cdot \dot{r}_j(0)} \]

\( \epsilon_d(m, \omega) \) is real: its zeros come as pairs of complex conjugates

\( \epsilon_d(m, \omega) = 0 \rightarrow \) polynomial of degree 2N: 2N zeros
Three derivations of Landau damping & growth by N-body mechanics

• A pedestrian, short, yet rigorous, one germane to Kaufman’s in a Vlasovian setting (1972)

Unambiguous mechanical understanding of Landau damping/growth: average synchronization of particles with the wave

• One showing waves damp because of phase mixing à la van Kampen (1955)

• One showing that the Vlasovian limit is singular and corresponds to a renormalized description of the actual N-body dynamics
Landau damping à la van Kampen-Dawson (1)

- 3D calculation analogous to Dawson’s 1960 one for a 1D plasma made up of many fluid monokinetic beams whose velocities are successive multiples of a small velocity $\delta$
- $\epsilon_d(m, \omega)$ brings two beam modes per beam
- Their eigenfrequencies are pairs of complex conjugate values for $\omega$, whose imaginary parts tend to vanish when $\delta$ decreases
- This makes these modes analogous to the van Kampen ones
Contour plots of the modulus of the granular dielectric function
Landau damping à la van Kampen-Dawson

- Landau damping recovered by phase mixing of the beam modes ("van Kampen modes")

- For the Landau instability, one of the beam modes becomes an unstable eigenmode with Landau growth rate

- However, the collaboration of this mode, of its damped companion, and of all remaining beam modes is necessary!

\[ e^{-i\omega_m t} \left( e^{\gamma_L(k_m)t} + e^{-\gamma_L(k_m)t} - e^{-\gamma_L(k_m)t} \right) = e^{-i\omega_m t + \gamma_L(k_m)t} \]
On average particles synchronize with Langmuir waves: enables cold beam-plasma instability
Free electron laser

 Beschleunigte Elektronen
 accelerated electrons

 Magnete
 magnets

 Elektronenauffänger
 electron dump

 Licht
 light

 Experiment
 experiment
Collisions at work

- Change of nature of the unstable modes when the velocity mismatch of beams $\delta$ decreases
- They go from individual beam-plasma ones to components of the collisional relaxation of the many-beam system toward a thermalized distribution, when it is slightly perturbed
- Their growth rate is mainly defined by global features of the plasma like the slope of $g(v)$ and depends weakly on the beam densities (see RMPP)
- Why?
Collisions at work

• Answer: Multi-beam-multi-arrays correspond to invariant states of the dynamics where neither collisions, nor waves are present.

• When perturbing the positions of particles of a multi-beam-multi-array having a dense enough set of velocities, they exponentially diverge from their initial positions, because of collisions.

• The finite value of the corresponding exponentiation rates is crucial for the equivalent of the van Kampen phase mixing to occur in the N-body system (see RMPP § 5.2).
Collisions matter in the calculation à la van Kampen

Dielectric function of the monokinetic beams whose velocities are successive multiples of a small velocity $\delta$

\[
\epsilon_{d1}(m, \omega) = 1 - \omega_p^2 \sum_{\sigma} \frac{g(\sigma \delta) \delta}{(\omega - \sigma k_m \delta)^2}.
\]

Zeros $\nu_{\sigma, \mu} = \alpha_\sigma + \mu i \beta_\sigma$, where $\mu = \pm 1$. 
Collisions matter in the calculation à la van Kampen

Dielectric function of the monokinetic beams whose velocities are successive multiples of a small velocity $\delta$

$$\epsilon_{d1}(m, \omega) = 1 - \omega_p^2 \sum_\sigma \frac{g(\sigma \delta) \delta}{(\omega - \sigma k_m \delta)^2}.$$ 

Zeros $\nu_{\sigma, \mu} = \alpha_\sigma + \mu i \beta_\sigma$, where $\mu = \pm 1$ bringing contributions

$$\Phi_{j m \sigma \mu}(r, t) = -\frac{e}{\varepsilon_0 k_m^2 L^3 \epsilon'_\sigma \mu} \frac{\exp[i(k_m \cdot (r - r_j(0)) - \nu_{\sigma, \mu} t)]}{\nu_{\sigma, \mu} - k_m \cdot \dot{r}_j(0)} + \text{c.c.}$$

where

$$\epsilon'_\sigma \mu = \frac{\partial \epsilon_{d1}}{\partial \omega}(m, \nu_{\sigma, \mu}) = 2\omega_p^2 \sum_\sigma \frac{g(\sigma k_m \delta) \delta}{(\nu_{\sigma, \mu} - \sigma k_m \delta)^3}.$$
The sum over \( j \) of the \( \Phi_{jm\sigma\mu}(r, t) \)'s yields

\[
\Phi_{m,\sigma,\mu}(r, t) = -\frac{\exp ik_m \cdot r}{\varepsilon_0 k_m^2 L^3} eN \int \frac{f(m, v)}{\nu_{\sigma,\mu} - k_m \cdot v} d^3v \frac{\exp(-i\nu_{\sigma,\mu}t)}{\epsilon'_{\sigma,\mu}} + c.c.,
\]

(55)

where \( f(r, v) = N^{-1} \sum_j \delta(r - r_j(0)) \delta(v - \dot{r}_j(0)) \), and \( f(m, v) \) is its spatial Fourier transform (\( \delta(\bullet) \) stands for the Dirac distribution).

\[
\nu_{\sigma,\mu} = \alpha_\sigma + \mu i \beta_\sigma, \text{ where } \mu = \pm 1.
\]
The sum over $j$ of the $\Phi_j m_\sigma \mu(r, t)$'s yields

$$\Phi_{m, \sigma, \mu}(r, t) = -\frac{\exp i k_m \cdot r}{\varepsilon_0 k_m^2 L^3} eN \int \frac{f(m, v)}{\nu_{\sigma, \mu} - k_m \cdot v} d^3v \frac{\exp(-i\nu_{\sigma, \mu} t)}{\varepsilon'_{\sigma, \mu}} + c.c.,$$

(55)

where $f(r, v) = N^{-1} \sum_j \delta(r - r_j(0)) \delta(v - v_j(0))$, and $f(m, v)$ is its spatial Fourier transform ($\delta(\bullet)$ stands for the Dirac distribution).

$$\nu_{\sigma, \mu} = \alpha_\sigma + \mu i \beta_\sigma,$$

where $\mu = \pm 1$.

The $\beta$'s scale like $\delta \ln(\delta/\nu T)$ (Appendix C of RMPP)

Therefore, the width of $\frac{f(m, v)}{\nu_{\sigma, \mu} - k_m \cdot v}$ in parallel velocities is larger than the mismatch $\delta$ of nearby beams: the integral over these velocities converges smoothly.

Since the $\beta$'s correspond to collisions, collisions bring this convergence.
Without any calculation: Landau damping cannot correspond to a damped eigenmode

• The $N$-body system is a Hamiltonian system
• ==> its Lyapunov exponents come in pairs: a damped eigenmode comes with a companion with the opposite growth rate (time-reversal symmetry)
• If Landau damping were due to a damped eigenmode, its growing companion would be excited for a typical initial condition
• Landau damping would not be visible!
Three derivations of Landau damping & growth by N-body mechanics

- A pedestrian, short, yet rigorous, one germane to Kaufman’s in a Vlasovian setting (1972)
  Unambiguous mechanical understanding of Landau damping/growth: average synchronization of particles with the wave
- One showing waves damp because of phase mixing à la van Kampen (1955)
- One showing that the Vlasovian limit is singular and corresponds to a renormalized description of the actual N-body dynamics
Fundamental equation for the electrostatic potential (2)

\[ \epsilon_d(m, \omega) \varphi(m, \omega) = \varphi^{(bal)}(m, \omega) \]

\[ k_m = \frac{2\pi}{L} m \]

\[ \epsilon_d(m, \omega) = 1 - \frac{\omega_p^2}{N} \sum_{j=1}^{N} \frac{1}{(\omega - k_m \cdot v_j)^2} \]

\[ \varphi^{(bal)}(m, \omega) = \sum_{j=1}^{N} \varphi_j^{(bal)}(m, \omega), \]

\[ \varphi_j^{(bal)}(m, \omega) = -\frac{ie}{\varepsilon_0 k_m^2} \frac{\exp[-ik_m \cdot r_j(0)]}{\omega - k_m \cdot \dot{r}_j(0)} \]

Substitute the **discrete sums** over particles with **integrals** over a smooth distribution function: singular limit
\[ \varepsilon_d(m, \omega) \varphi(m, \omega) = \varphi^{(bal)}(m, \omega) \]

\[ k_m = \frac{2\pi}{L} m \quad \varepsilon_d(m, \omega) = 1 - \frac{\omega_p^2}{N} \sum_{j=1}^{N} \frac{1}{(\omega - k_m \cdot v_j)^2} \]

\[ \varepsilon(k_m, \omega) \varphi(k_m, \omega) = \varphi^{(bal)}(k_m, \omega) \]

\[ \varepsilon(k_m, \omega) = 1 - \frac{\omega_p^2 L^3}{N} \int \frac{f_0 \cdot v}{(\omega - k_m \cdot v)^2} d^3v. \]

Fourier transform of \( \delta f \)

\[ \varphi^{(bal)}(k_m, \omega) = -\frac{i\varepsilon}{\epsilon_0 k_m^2} \int \frac{\delta f(k_m, v, t = 0)}{\omega - k_m \cdot v} d^3v, \]

\[ \varphi^{(bal)}(m, \omega) = \sum_{j=1}^{N} \varphi_j^{(bal)}(m, \omega), \]

Fourier transform of \( \delta[r - r_j(0)] \)

\[ \varphi_j^{(bal)}(m, \omega) = -\frac{i\varepsilon}{\epsilon_0 k_m^2} \exp[-ik_m \cdot r_j(0)] \frac{\exp[-ik_m \cdot r_j(0)]}{\omega - k_m \cdot \dot{r}_j(0)} \]
Vlasovian linear equation for the electrostatic potential recovered by taking a singular limit

\[ \epsilon(k_m, \omega) \varphi(k_m, \omega) = \varphi^{(bal)}(k_m, \omega), \]

\[ \epsilon(k_m, \omega) = 1 - \frac{\omega_p^2 L^3}{N} \int \frac{f_0(v)}{(\omega - k_m \cdot v)^2} \, d^3v. \]

\[ \varphi^{(bal)}(k_m, \omega) = -\frac{ie}{\epsilon_0 k_m^2} \int \frac{\delta f(k_m, v, t = 0)}{\omega - k_m \cdot v} \, d^3v, \]
Fundamental equation for the electrostatic potential (2)

\[ \epsilon_d(m, \omega) \varphi(m, \omega) = \varphi^{(\text{bal})}(m, \omega) \]

\[ k_m = \frac{2\pi}{L} m \]

\[ \epsilon_d(m, \omega) = 1 - \frac{\omega_p^2}{N} \sum_{j=1}^{N} \frac{1}{(\omega - k_m \cdot v_j)^2} \]

\[ \varphi^{(\text{bal})}(m, \omega) = \sum_{j=1}^{N} \varphi_j^{(\text{bal})}(m, \omega), \]

\[ \varphi_j^{(\text{bal})}(m, \omega) = -\frac{ie}{\varepsilon_0 k_m^2} \frac{\exp[-ik_m \cdot r_j(0)]}{\omega - k_m \cdot \dot{r}_j(0)} \]

Substitute the **discrete sums** over particles with **integrals** over a smooth distribution function: singular limit
One step back: keep discrete sums over particles for the initial conditions

\[ \epsilon(k_m, \omega) \varphi(k_m, \omega) = \varphi^{(\text{init})}(k_m, \omega), \]

\[ \epsilon(k_m, \omega) = 1 - \frac{\omega_p^2 L^3}{N} \int \frac{f_0(v)}{(\omega - k_m \cdot v)^2} \, d^3v \]

\[ \varphi^{(\text{init})}(k_m, \omega) = -\frac{ie}{\epsilon_0 k_m^2} \sum_{j=1}^{N} \exp[-ik_m \cdot r_{j0}] \frac{\exp[-i(k_m \cdot v_j) \omega]}{\omega - k_m \cdot v_j} \]

In the inverse Laplace transform of \( \varphi(k_m, \omega) \) contribution of the **zeros** of \( \epsilon(k_m, \omega) \) (Vlasovian value)
And of the **poles** of \( \varphi^{(\text{init})}(k_m, \omega) \)

Solution in textbooks (\( N=1 \) particle + Vlasovian plasma):
- Landau poles defining damped or growing Langmuir waves
- Debye shielded electrostatic potential of particles
Mysterious term of initial conditions

In the Vlasovian picture:

• Term of initial conditions + Landau poles

\[ \varepsilon(m, \omega) \Phi(m, \omega) = \phi^{(bal)}(m, \omega) \]

\[ \Phi^{(bal)}(m, \omega) = -\frac{ie}{\epsilon_0 k_m^2} \int \frac{f(m, v)}{\omega - k_m \cdot v} \, d^3v \]

• + shielded potentials

Term of initial conditions = continuous limit of all shielded potentials: physical meaning?
New insight into the Vlasovian limit/1

1. It is singular

- When increasing the number of beams, the number of zeros of $\varepsilon_d(m, \omega)$ keeps increasing, but is fixed for $\varepsilon(m, \omega)$

- Transition from a finite increasing number of poles for $\varepsilon_d(m, \omega)$ to a cut for $\varepsilon(m, \omega)$

\[
\varepsilon_d(m, \omega) = 1 - \frac{\omega_p^2}{N} \sum_{j=1}^{N} \frac{1}{(\omega - k_m \cdot v_j)^2}
\]

\[
\varepsilon(m, \omega) = 1 - \omega_p^2 \int \frac{f_0(v)}{(\omega - k_m \cdot v)^2} \, d^3v
\]

\[
= 1 + \frac{\omega_p^2}{k_m^2} \int \frac{k_m \cdot \nabla_v f_0(v)}{(\omega - k_m \cdot v)} \, d^3v
\]
New insight into the Vlasovian limit/2

• Mechanically, Landau damping is due to a phase mixing, and Landau growth involves a phase mixing too; this latter fact is absent in the Vlasovian approach

Some memory though: instability involves an eigenmode, damping involves analytic continuation

The wave echo experiment proved the existence of the beam modes in a genuine (granular) plasma

Baker, Ahern, & Wong, 1968
New insight into the Vlasovian limit/3

• It is impossible to give a physical interpretation to the term of initial conditions in Landau's calculation of Langmuir waves.
  It is nothing but the continuous limit of the sum of the ballistic potentials of the N electrons.

• Adding a test particle to the N-body system does not provide the shielded potential of this particle, in contrast with the Vlasovian case.
New insight into the Vlasovian limit/4

2. It corresponds to a renormalized description of the actual N-body dynamics

• One of the simplest examples of a renormalized potential is the Debye shielded potential
  It is a mean-field potential found when adding a test particle to a Vlasovian plasma: “dressed particle”

• Clarified after explaining Debye shielding mechanically

• Mean-field and BBGKY derivations show that Vlasov equation deals with a mean-field potential
New insight into the Vlasovian limit/5

• Therefore, the Vlasovian dielectric function is a renormalized version of that of a multi-beam-multi-array

• A Vlasovian Langmuir wave is the renormalized version of a set of beam modes of the N-body system: a wave with “dressed particles”
From N-body dynamics to Debye shielding and Landau damping

N body → mean field

BBGKY → Vlasov → linearized → Landau damp.

2 point f → fluid equations

+ Boltzmann equil.

+ test particle

Textbooks

New theory

N body → Debye shielding & Landau damping

First computing N-body dynamics, then introducing \( f(v) \)

No probabilistic argument, no PDE
Explicit derivation of the fundamental equation for the electrostatic potential
Standard Vlasovian derivation of Landau damping

Uniform ionic neutralizing background + Vlasovian electrons with perturbed \( f(r,v,t) = f_0(v) + \delta f(r,v,t) \)

Calculation:

• Linearization of Vlasov equation
• Fourier-Laplace transform
• Yields a linear equation for the electrostatic potential

\[ \epsilon(k_m, \omega) \varphi(k_m, \omega) = \varphi^{(\text{init})}(k_m, \omega) \]

• Landau damping obtained from the zero of when going back to the \( t \) dependent quantities

The N-body derivation parallels the Vlasovian one
N-body plasma

- Infinite plasma with spatial periodicity $L$ in three orthogonal directions
- Made up of $N$ electrons in each elementary cube with volume $L^3$
- Uniform ionic neutralizing background
Distributions of particles

- Almost uniform distributions of particles: set of monokinetic beams; each beam is a simple cubic array of particles

3D (x,y,z) picture

1D (x,v) picture

\[ E=0 \]
Fundamental equation for the electrostatic potential (1)

- Perturbation from ballistic orbits of electrons
  \[ r = r_0 + v_0 t + \delta r \]

- Laplace transform the perturbed Coulomb potential

- Laplace transform Newton equation for the particles acted upon by this potential

- Combine these equations
Principle of the derivation provided in three steps:

• For a toy model with a single particle and a single Fourier component

• One inductive step to the genuine N-body linear equation for the full electrostatic potential

• Use that particles sit on spatial arrays: big cancellations!

• Needs enough beam particles

• Next 3 viewgraphs
Toy model with a single particle and a single Fourier component

The particle:

• produces a single harmonic of the Fourier decomposition of the electric field due to an electron

• is submitted to this field

\[ \Delta \phi = -\frac{e}{\varepsilon_0} \delta(x - r) \]  

Poisson equation

\[ \tilde{\phi}(t) = k^{-2} \exp[-i k \cdot r(t)] \]  

Fourier component of the potential due to the particle at \( r(t) \) with appropriate normalization

Principle of the derivation on next viewgraph
\[ \tilde{\phi}(t) = k^{-2} \exp[-i k \cdot r(t)] \]
\[ r(t) = r_0 + vt + \Delta r(t) \quad \Delta r(t) \text{ mismatch to ballistic motion} \]
\[ \tilde{\phi}(t) = k^{-2} \exp[-i k \cdot (r_0 + vt)] \left[ 1 - i k \cdot \Delta r(t) \right]. \quad \text{Linearization in } \Delta r \]
\[ \phi(\omega) = k^{-2} \exp[-i k \cdot r_0] \left[ \frac{1}{\omega - k \cdot v} - i k \cdot \Delta r(\omega - k \cdot v) \right] \quad \text{Laplace transformed} \]
\[ \ddot{r} = \nabla \phi(r) \quad \text{Newton’s equation with appropriate normalization} \]
\[ \phi(R) = \text{coefficient } \tilde{\phi} \exp(i k \cdot R) \quad \text{Potential in real space} \]
\[ \Delta \ddot{r} = \text{constant } k \tilde{\phi}(t) \exp[i k \cdot (r_0 + vt)] \]
\[ -\omega^2 \Delta r(\omega) = \text{constant } k \exp(i k \cdot r_0) \phi(\omega + k \cdot v) \]

Laplace transformed

\[ k^2 \phi(\omega) - \omega_p^2 k \cdot k \frac{\phi(\omega + k \cdot v - k \cdot v)}{(\omega - k \cdot v)^2} \exp[i(k - k) \cdot r_0] \]
\[ = k^2 \phi^{(\text{bal})}(\omega), \quad \phi^{(\text{bal})}(\omega) = k^{-2} \exp[-i k \cdot r_0] \frac{\exp[-i k \cdot r_0]}{\omega - k \cdot v} \]
For any finite $\Delta r$, linearization correct for finite $k$'s only

Truncate the Coulomb potential at a scale $b_{\text{smooth}}$ much smaller that the Debye length; $m$ such that $k_m b_{\text{smooth}} < 1$, $b_{\text{smooth}} \ll \lambda_D$

useful scales

In the real derivation terms with $n =/= m$ killed by « destructive interference »: beam particles sit on arrays $\epsilon_d(m, \omega) \varphi(m, \omega) = \varphi^{(\text{bal})}(m, \omega)$
Conclusion of the three derivations

• Technically: the continuous velocity distribution is introduced after particle dynamics has been taken into account, and not before, as occurs in the kinetic approach

• Three different views of Landau damping:
  - A pedestrian though rigorous calculation à la Kaufman derives Landau damping and proves it corresponds to an average synchronization of close to resonant particles
  - A fundamental equation for the electro-static potential recovers the analogs of both van Kampen and Landau theories performed in a Vlasovian setting
The phase mixing of many beam modes produces Landau damping, which cannot correspond to a damped eigenmode because of Hamiltonian mechanics.

This phase mixing is also active for Landau growth.

The Vlasovian limit is singular and corresponds to a renormalized description of Coulomb interactions.

\[ \gamma_L(k_m) = \frac{\pi \omega_p^2}{k_m^2} \frac{\partial^2 \varepsilon_r}{\partial \omega} g'(\frac{\omega_{mr}}{k_m}) \]

Landau damping/growth important for many waves in magnetic confinement, solar wind, etc...
Feynman’s Nobel lecture:
``Perhaps a thing is simple if you can describe it fully in several different ways without immediately knowing that you are describing the same thing."
Open issues

• Extension to magnetized plasmas?
• Would quantum mechanics bring a different view?

Would Vlasovian continuity be closer to the truth?

See Appendix D of RMPP
References/1


Particles with velocity $u$ at $t = 0$ and uniform initial spatial distribution in a wave

$E(x, t) = E_w \cos(x - \omega t)$

1. **Elementary calculation**

Equation of motion $\ddot{x} = \varepsilon \cos x$ in the frame of the wave

$\varepsilon = -eE_w/m$

Unperturbed orbit is $X_0(t) = x_0 + ut$

Complementing with arbitrary polynomial terms up to power 5 in $t$

Equating terms of degree up to 3 yields the beginning of Taylor expansion of $x(t)$

Averaging over $x_0$

$$\Delta u(t) \equiv \langle \dot{x}(t) \rangle - u = -\frac{\varepsilon^2 u t^4}{24}. \quad (1)$$
\[ \Delta u(t) \equiv \langle \dot{x}(t) \rangle - u = -\frac{\varepsilon^2 u t^4}{24}. \]

\( u\Delta u(t) < 0 \Rightarrow \) average synchronization of the particles with the wave for \( t \) small Vanishing effect for small \(|u|\)'s \Rightarrow \) effect not related to trapping
2. Better calculation

Second order perturbation in $\varepsilon$

Picard’s fixed point method: a very efficient way to perform the calculation

$$X_n \rightarrow X_{n+1}$$

$$\ddot{X}_{n+1} = \varepsilon \cos X_n,$$

with the boundary conditions $X_n(0) = x_0$ and

$$\dot{X}_n(0) = u$$

$$X_1 = x_0 + ut - \varepsilon \sin(x_0)t/u - \varepsilon [\cos(X_0(t)) - \cos(x_0)]/u^2$$

Identical to the result of first order perturbation theory

But obtained without writing a linearized version of the equation of motion
$X_2$ to second order in $\varepsilon$: result of first order perturbation theory without writing a second order expression of the equation of motion

$$
\Delta u(t) = \varepsilon^2 \frac{\cos ut - 1 + \frac{1}{2} ut \sin (ut)}{u^3}.
$$

$u\Delta u(t)$ even in $t$
negative from $t = 0$ up to $t = T \equiv 2\pi/|u|$: synchronization

$\Delta u(t) \sim 1/u^3 \Rightarrow$ average synchronization small for large $|u|$’s

Doveil, DFE, and Macor


Calculation in Nicholson’s textbook