Without any calculation: Landau damping cannot correspond to a damped eigenmode

- The $N$-body system is a Hamiltonian system
- $\Rightarrow$ its Lyapunov exponents come in pairs: a damped eigenmode comes with a companion with the opposite growth rate (time-reversal symmetry)
- If Landau damping were due to a damped eigenmode, its growing companion would be excited for a typical initial condition
- Landau damping would not be visible!

**Liouville theorem:** keep area constant
Smoothed potential
Cutoff at small scales

$$\varphi(r) = \frac{1}{L^3} \sum_{m, k_m b_{smooth} \leq 1} \tilde{\varphi}(m) \exp(ik_m \cdot r).$$

$$k_m = \frac{2\pi}{L} m, \quad b_{smooth} \ll \lambda_D \quad \lambda_D = \frac{v_T}{\omega_p} = \left[\frac{\varepsilon_0 T}{ne^2}\right]^{1/2}$$
Collisions are present despite the cutoff

Impact parameter b

\[ \lambda_{id} \ll b_{smooth} \]

\[ b_{smooth} \ll \lambda_D \]

\[ \lambda_{ca} \]

\[ 10^{-10} \quad 10^{-5} \]

Distance [m]

\[ n = 10^{19} \text{ m}^{-3} \quad T = 1 \text{ keV} \]

\[ \lambda_{ca} = \frac{e^2}{4\pi \varepsilon_0 T} \]

\[ \lambda_{id} = n^{-1/3} \quad \lambda_D = \frac{v_T}{\omega_p} = \left[ \frac{\varepsilon_0 T}{ne^2} \right]^{1/2} \]
Debye shielding: a consequence of collisions
Unification of Landau damping and spontaneous emission

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Debye shielding: a consequence of collisions

Unification of Landau damping and spontaneous emission
What is Debye shielding for a typical particle in a plasma?

\[ n_e(x) = n_0 \exp \left( \frac{e \varphi(x)}{k_B T_e} \right) \]

Intuitive: A Langmuir probe can organize the plasma particles in its vicinity to produce a sheath which shields it at large distances.

This intuition carries over to a test particle in a plasma.

What about the shielding of any plasma particle by the other ones? [Gasiorowicz et al. 1956, Balescu 1963, Rostoker 1964]

How can any particle help shielding all the other ones while being shielded by the latter?

How can each particle be shielded by all other ones, while all the plasma particles are in uninterrupted motion?
Plasma as a pea soup

P. H. Diamond, S.-I. Itoh and K. Itoh, Modern Plasma Physics, 2009 p. 34

"the basic picture of an equilibrium plasma is one of a soup or gas of dressed test particles. In this picture, each particle:

i.) stimulates a collective response from the other particles by its discreteness

ii.) responds to and ‘dresses’ other discrete particles by forming part of the background Vlasov fluid.

Thus, if one views the plasma as a pea soup, then (...) ‘each pea in the soup acts like soup for all the other peas' "
Fig. 2.2. A large number of particles exist within a Debye sphere of particle ‘A’ (shown by black) in (a). Other particles provide a screening on the particle ‘A’. When the particle ‘B’ is chosen as a test particle, others (including ‘A’) produce screening on ‘B’, (b). Each particle acts the role of test particle and the role of screening for other test particle.
Debye shielding: a mere consequence of collisions

• Intuitive picture
• Derivation in the logics of N-body Coulomb interaction
Part of the global deflection due to a single particle

The apparent charge of the red particle decreases when the radius increases with respect to the case with no Coulomb repulsion.

A particle with impact parameter $b$ must travel a distance $\sim b$ to be fully deflected. This takes a time $t = b/\nu_{th}$.

Even fewer electrons in the purple circle because of Coulomb repulsion.

Less electrons in the black circle because of Coulomb repulsion.
Intuitive interpretation of shielding
In written words
Uniformly distributed particles at $t = 0$
Single out particle $P$ •
Deflection by $P$ of a particle with impact parameter $b$ completed after this particle travels a distance several $b$’s
At time $t$, $P$ has deflected all particles with a typical $b$ of the order or less than $v_{th} t$
This part of their global deflection due to particle $P$ reduces the number of particles inside the sphere $S_P(t)$ of radius $v_{th} t$ about it
According to Gauss’ theorem, the effective charge of particle $P$ as seen out of $S_P(t)$ is reduced:

the charge of particle $P$ is shielded due to these deflections

Shielding effect increases with $t$

$\Rightarrow$ increases with the distance to particle $P$

The deflections due to P are compensated by those of the other electrons

Global deflection of particles due to many other ones

$\Rightarrow$ the density of the electrons does not change

At variance with the shielding at work next to a probe (Piel 2010)
Typical time-scale for shielding to set in: time for a thermal particle to cross a Debye sphere, i.e. $\omega_p^{-1}$.

Shielding, though very fast a process, is a cooperative dynamical one, not a collective or coherent one:

results from the accumulation of almost independent repulsive deflections having the same qualitative impact on the effective electric field of particle $P$.

If point-like ions were present, the attractive deflection of charges with opposite signs would have the same effect.

Shielding and collisional transport are two aspects of the same two-body repulsive process!
Debye shielding: self-organization by “Collisions”

How a particle can be shielded by all other ones, while contributing to shielding them all? Reply: the Coulombian deflections of electrons by a given electron P decrease the number of electrons about P, which decreases its apparent charge, according to Gauss' theorem: dressed particle

Debye shielding results from a cooperative dynamical self-organization process, produced by the accumulation of almost independent Coulomb deflections over a plasma period

Particles slightly deflect each other to generate a Debye-shielded interaction
“Collisions” bring self-organization to plasmas: Debye shielding

Pictorially: while electron P repels more and more electrons when the distance from P increases, its repulsion progressively vanishes. This repulsion vanishes at about the Debye length. This idea is used explicitly to compute Debye shielding of a probe in [Meyer-Vernet 1993].

Debye shielding and collisional transport are two sides of the same coin: Coulomb deflections. These deflections bring both order and chaos.
• Coulombian deflections are traditionally viewed as the cause of a disorder leading to collisional transport

• N-body approach: the cooperative action of these deflections produces order too

• After one plasma period typically, this action makes the plasma to behave as a dielectric, and Debye shielding to set in

• One plasma period typically the time necessary for phase mixing

• Paradoxically, the Vlasovian dielectric behavior turns out to exist thanks to what is usually called "collisions"! Langmuir wave with dressed particles
Almost uniform distributions of particles (Cf. Vlasovian $f_0(\mathbf{v})$): set of monokinetic beams; each beam is a simple cubic array of particles.
Debye shielding: a consequence of collisions

• Intuitive picture

• Derivation in the logics of N-body Coulomb interaction
Shielding made intuitive through N-body dynamics

Dynamics of the same system of N electrons

Newton’s second law for electrons

Picard iteration technique for $\frac{d\mathbf{X}}{dt} = f(\mathbf{X})$: $\frac{d\mathbf{X}_{n+1}}{dt} = f(\mathbf{X}_n)$

Start with $\mathbf{X}_0$ some approximation of true orbit; $n$ goes to infinity

Rigorous calculation describing the true dynamics

No statistical setting

Uncovers: The acceleration of particle A due to particle B has a part mediated by all other particles, which shields particle B

This results from the deflection of all other particles by particle B

This deflection modifies the force on particle A

$\Rightarrow$ lower apparent charge of B by Gauss’ theorem

One plasma period necessary for shielding to occur

Cooperative effect: self-organization!

Debye shielding and collisional transport are two sides of

Coulomb deflections: self-organization and transport
Advertisement for Picard iteration technique

• Simple Picard calculation: last two viewgraphs of « Landau damping as phase mixing Singular Vlasovian description Synchronization »

• Very efficient to simplify perturbation calculations
Derivation of Debye shielding using the Laplace transform: no intuitive picture of this effect 
Intuitive picture provided by Picard technique for the dynamics in real time of one particle 
in the full OCP Coulomb potential due to the 
\( N - 1 \) other particles

\[
\ddot{r}_l = \frac{e}{m_e} \nabla \varphi_l(r_l), \tag{1}
\]

\[
\varphi_l(r) = \frac{1}{L^3} \sum_m \tilde{\varphi}_l(m) \exp(ik_m \cdot r). \tag{2}
\]

\[
\tilde{\varphi}_l(m) = -\frac{e}{\varepsilon_0 k^2_m} \sum_{j=1, j \neq l}^{N} \exp(-ik_m \cdot r_j), \tag{3}
\]
Picard iteration technique

$$\dot{r}_l^{(n)} = \frac{e}{m_e} \nabla \varphi_l(r_l^{(n-1)})$$  \hspace{1cm} (4)

Zeroth order : ballistic approximation

$$r_j^{(0)} = r_{j0} + v_j t$$  \hspace{1cm} (5)

First Picard iterate identical to first order perturbation theory

Actual orbit corresponds to \( n \rightarrow \infty \)
\[ \xi_l^{(n)} = r_l^{(n)} - r_l^{(0)} \] mismatch with respect to the ballistic orbit at the \( n \)-th iterate \( \xi_i^{(0)} = 0 \) and \( \dot{\xi}_i^{(0)} = 0 \) for all \( i \)'s

Picard's \( n \)-th iterate

\[
\ddot{\xi}_l^{(n)} = \sum_{j=1, j \neq l}^{N} \ddot{\xi}_{lj}^{(n)},
\]

\[
\ddot{\xi}_{lj}^{(n)} = a_C(r_l^{(n-1)} - r_j^{(n-1)})
\]

\( \xi_{lj}^{(n-1)} \): deflection of particle \( l \) by particle \( j \)

\[
a_C(r) = \frac{i e^2}{\epsilon_0 m_e L^3} \sum_{m \neq 0} k_m^{-2} k_m \exp(ik_m \cdot r).
\]

\( \xi_{lj}^{(n-1)} \) anti-symmetrical in \((l, j)\) (action-reaction)
Taylor expansion of $\dddot{\xi}_{l,j}^{(n)}$ in $a$, order of magnitude of the total Coulombian acceleration

$$\dddot{\xi}_{l,j}^{(n)} = \sum_{j=1, j \neq l}^{N} \left[ \dddot{\xi}_{l,j}^{(1)} + M_{l,j}^{(n-1)} + 2\nabla a_{C}(r_{l}^{(0)} - r_{j}^{(0)}) \cdot \xi_{l,j}^{(n-1)} \right] + O(a^3),$$

$$M_{l,j}^{(n-1)} = \nabla a_{C}(r_{l}^{(0)} - r_{j}^{(0)}) \cdot \left[ \xi_{l}^{(n-1)} - \xi_{j}^{(n-1)} - 2\xi_{l,j}^{(n-1)} \right]$$

$$= \nabla a_{C}(r_{l}^{(0)} - r_{j}^{(0)}) \cdot \sum_{i=1, i \neq l,j}^{N} (\xi_{li}^{(n-1)} - \xi_{ji}^{(n-1)})$$

$$= \nabla a_{C}(r_{l}^{(0)} - r_{j}^{(0)}) \cdot \sum_{i=1, i \neq l,j}^{N} (\xi_{li}^{(n-1)} + \xi_{ij}^{(n-1)}),$$

Particle $j$ modifies the motion particle $i \Rightarrow$ action of particle $i$ on particle $l$ modified by particle $j \Rightarrow$ Coulomb acceleration of ballistic particle $l$ due to ballistic particle $j$ modified by the mediation of particle $i$. 
\[
\dddot{\xi}_l^{(n)} = \sum_{j=1,j\neq l}^N \left[ \dddot{\xi}_{lj}^{(1)} + M_{lj}^{(n-1)} + 2 \nabla a_C(\mathbf{r}_l^{(0)} - \mathbf{r}_j^{(0)}) \cdot \dddot{\xi}_{lj}^{(n-1)} \right] + O(a^3),
\]

\[
M_{lj}^{(n-1)} = \nabla a_C(\mathbf{r}_l^{(0)} - \mathbf{r}_j^{(0)}) \cdot \left[ \xi_l^{(n-1)} - \xi_j^{(n-1)} - 2 \xi_{lj}^{(n-1)} \right]
\]

\[
= \nabla a_C(\mathbf{r}_l^{(0)} - \mathbf{r}_j^{(0)}) \cdot \sum_{i=1,i\neq l,j}^N \left( \xi_{li}^{(n-1)} - \xi_{ji}^{(n-1)} \right)
\]

\[
= \nabla a_C(\mathbf{r}_l^{(0)} - \mathbf{r}_j^{(0)}) \cdot \sum_{i=1,i\neq l,j}^N \left( \xi_{li}^{(n-1)} + \xi_{ij}^{(n-1)} \right),
\]

\[M_{lj}^{(n-1)}: \text{acceleration of particle } l \text{ due to particle } j \text{ mediated by all other particles}\]

Both particles \( j \) and \( l \) shifted with respect to their ballistic positions \( \Rightarrow \text{last term in the bracket} \)
Uncovers mechanical background of the calculation of shielding using the equilibrium pair correlation function:
shielding results from the correlation of two particles occurring through the deflective action of all the other ones
Compound effect of the deflection of particle \( i \) by particle \( j \): smaller negative charge inside a sphere centered on particle \( j \)
New insight into the Vlasovian limit/4

2. It corresponds to a renormalized description of the actual N-body dynamics

- One of the simplest examples of a renormalized potential is the Debye shielded potential
  It is a mean-field potential found when adding a test particle to a Vlasovian plasma

- Will be made clearer after explaining Debye
  The acceleration of particle A due to particle B has a part mediated by all other particles, which shields particle B
  The Debye shielded potential is a renormalized potential: that of a dressed particle
Conclusion for « Debye shielding »

• The N-body approach shows that collisions play an essential role in collisionless plasmas

• Lecture #2: The finite value of the exponentiation rates of beam modes is crucial for the equivalent of the van Kampen phase mixing to occur in the N-body system (see section 5.6 of RMPP)

• Lecture #3: Debye shielding is a direct consequence of collisions

• Coulomb collisions, usually viewed as the cause of a disorder, in particular of Coulomb scattering and of collisional transport, induce order too by their cooperative action
Open issues

• Debye shielding perpendicularly to a magnetic field?
• ???
References


Debye shielding: a consequence of collisions

Unification of Landau damping and spontaneous emission

with an extension to quasilinear theory
When dealing with particles, spontaneous emission comes about naturally
Reference case: Cherenkov emission in a material medium
It is an incoherent mechanism: a statistical description is mandatory
This emission is easier to describe in a setting where both waves and particles are present
Setting provided by a systematic reduction of the number of degrees-of-freedom of the N-body system retaining only the collective vibrations of the bulk plasma, but keeping a granular description of the tail particles
Summary

• From N-body dynamics to wave-particle interaction
• Unification of Landau damping and spontaneous emission
From N-body dynamics to wave-particle interaction

Retain only the collective vibrations of bulk particles (Langmuir waves) coupled with $N' \ll N$ tail particles:
- An amplitude equation is derived for each wave where tail particles provide a source term.
- Newton equations for each tail particle acted upon by the waves

Defines the self-consistent dynamics of $M$ waves with $N'$ tail particles
Keep M waves only
Self-consistent Hamiltonian

\[ H_{sc} = \sum_{r=1}^{N} \frac{p_r^2}{2} + \sum_{j=1}^{M} \omega_{j0} \frac{X_j^2 + Y_j^2}{2} \]

+ \varepsilon \sum_{r=1}^{N} \sum_{j=1}^{M} k_j^{-1} \beta_j (Y_j \sin k_j x_r - X_j \cos k_j x_r) \quad (11)

\[ H_{sc} = \sum_{r=1}^{N} \frac{p_r^2}{2} + \sum_{j=1}^{M} \omega_{j0} I_j - \varepsilon \sum_{r=1}^{N} \sum_{j=1}^{M} k_j^{-1} \beta_j \sqrt{2I_j} \cos(k_j x_r - \theta_j) \]

\[ \beta_j = \omega_p \left( \frac{\partial \varepsilon}{\partial \omega} (k_j, \omega_j) \right)^{-1/2} \quad (12) \]

\[ \varepsilon = \sqrt{\frac{2}{N_{\text{bulk}}}} \quad (13) \]
\begin{align*}
\dot{x}_r &= p_r \\
\dot{p}_r &= -\varepsilon \sum_{j=1}^{M} \beta_j \sqrt{2I_j} \sin(k_j x_r - \theta_j)
\end{align*}

\begin{align*}
\dot{\theta}_j &= \omega_j - \varepsilon \sum_{r=1}^{N} k_j^{-1} \beta_j (2I_j)^{-1/2} \cos(k_j x_r - \theta_j) \\
\dot{I}_j &= \varepsilon \sum_{r=1}^{N} k_j^{-1} \beta_j (2I_j)^{1/2} \sin(k_j x_r - \theta_j)
\end{align*}

$H_{sc}$ invariant by $x_r \mapsto x_r + \alpha$, $\theta_j \mapsto \theta_j + k_j \alpha$

Total momentum $P_{sc} = \sum_{r=1}^{N} p_r + \sum_{j=1}^{M} k_j I_j$ is conserved
Summary

• From N-body dynamics to wave-particle interaction
• Unification of Landau damping and spontaneous emission
\[ \dot{x}_r = p_r \]
\[ \dot{p}_r = -\varepsilon \sum_{j=1}^{M} \beta_j \sqrt{2I_j} \sin(k_j x_r - \theta_j) \]
\[ \dot{\theta}_j = \omega_j - \varepsilon \sum_{r=1}^{N} k_j^{-1} \beta_j (2I_j)^{-1/2} \cos(k_j x_r - \theta_j) \]
\[ \dot{I}_j = \varepsilon \sum_{r=1}^{N} k_j^{-1} \beta_j (2I_j)^{1/2} \sin(k_j x_r - \theta_j) \]

Second order perturbation analysis in \( \varepsilon \):
Zeroth order in \( \varepsilon \):
uncoupled motion of waves and particles
Second order equation for the wave action
Averaged over initial positions $x_{l0}$

$$
\langle \hat{I}_{ij}^{(2)} \rangle = - \sum_{l=1}^{N} \varepsilon^2 \beta_j^2 I_{j0} \frac{\partial}{\partial \Omega_{lj}} \left( \frac{\sin \Omega_{lj} t}{\Omega_{lj}} \right) + \sum_{l=1}^{N} \varepsilon^2 \beta_j^2 k_j^{-2} \frac{\sin \Omega_{lj} t}{\Omega_{lj}}.
$$

(20)

Go to the continuum limit for the particle distribution

$$
\Omega_{lj} = k_j p_{l0} - \omega_{j0}
$$

$$
\varphi_{lj} = k_j x_{l0} - \theta_{j0}
$$
Integrating by parts

\[ \langle \hat{I}_j^{(2)} \rangle = N \varepsilon^2 \beta_j^2 I_{j0} \int \frac{\sin \Omega_j t}{\Omega_j} \frac{df}{dp} \frac{\partial p}{\partial \Omega_j} \frac{d\Omega_j}{k_j} \]

\[ + N \varepsilon^2 \beta_j^2 k_j^{-2} \int \frac{\sin \Omega_j t}{\Omega_j} f(p) \frac{d\Omega_j}{k_j} \]

\[ p = (\omega_{j0} + \Omega_j)/k_j \]
$\Delta p_f$ characteristic scale of variation of $f(p)$

Assume

$$t \gg \frac{2\pi}{k_j \Delta p_f}$$

$$\langle \dot{I}_j \rangle = 2\gamma_{jL} I_{j0} + S_j$$  \hspace{1cm} (24)

$$\gamma_{jL} = \alpha_j \frac{df}{dp}(\omega_{j0}/k_j)$$  \hspace{1cm} (25)

$$S_j = \frac{2\alpha_j}{k_j} f(\omega_{j0}/k_j)$$  \hspace{1cm} (26)

$$\alpha_j = \frac{\pi}{2} N \varepsilon^2 \beta_j^2 k_j^{-2}.$$  \hspace{1cm} (27)

Landau damping and spontaneous emission

Verified in numerical simulations (Doxas and Cary, 1997)
If \( df/dp < 0 \)

\[
I_{js} = -\frac{f(\omega_j/\kappa_j)}{\kappa_j f'(\omega_j/\kappa_j)}
\]  \( (28) \)

Damping: exponential relaxation to the “thermal” level

Intensity far above this level: looks like exponential damping

“Thermal” level vanishes in the limit \( N \to \infty \)

A mode under this level increases till it reaches the “thermal” level

For \( df/dp > 0 \): exponential growth
Reaching the thermal level

\[
\begin{align*}
I_j &> I_{js} \\
\frac{\partial f}{\partial p} < 0 \\
\frac{\partial f}{\partial p} > 0
\end{align*}
\]
Similar perturbative calculation for particles

\[
\langle \dot{p}_l \rangle = \varepsilon^2 \sum_{j=1}^{M} \beta_j k_j [I_j \frac{\partial}{\partial \Omega_{l_j}} \left( \frac{\sin \Omega_{l_j} t}{\Omega_{l_j}} \right) - \frac{1}{k_j^2} \frac{\sin \Omega_{l_j} t}{\Omega_{l_j}}] \tag{29}
\]

\[
\langle \dot{I}_j \rangle = \varepsilon^2 \sum_{l=1}^{N} \beta_j^2 \left[ -I_j \frac{\partial}{\partial \Omega_{l_j}} \left( \frac{\sin \Omega_{l_j} t}{\Omega_{l_j}} \right) + \frac{1}{k_j^2} \frac{\sin \Omega_{l_j} t}{\Omega_{l_j}} \right] \tag{30}
\]

First term linked with Landau growth and damping
Second term linked with spontaneous emission
Average total momentum conserved

\[ \sum_{l=1}^{N} \langle \dot{p}_l \rangle + \sum_{j=1}^{M} k_j \langle \dot{I}_j \rangle = 0, \quad (31) \]

Figure 4.2. Average Landau force acting upon a particle versus \( \Omega_{lj} t \)

Vanishes for a particle resonating with the wave
Mainly positive (resp. negative) for particles slower (resp. faster) than the wave: synchronization mechanism involves velocities close to the phase velocity of the wave.

Width in $pl - \omega_{j0}/k_j$ about $1/(k_j t)$

$1/|\gamma_{jL}|$ time related to Landau effect

⇒ synchronization efficient in a range $|\gamma_{jL}|/k_j$

about $\omega_{j0}/k_j$

Resonance interval about the phase velocity

Interval much larger than the wave trapping width:

weak resonance, trapping negligible
Going to the continuous spectrum limit

\[
\frac{\langle \Delta p^2 \rangle}{2 \Delta t} = \frac{L \varepsilon^2 \beta^2}{2p} I(p) = D_{QL}(p) \quad (32)
\]

\[
\frac{\langle \Delta p \rangle}{\Delta t} = \langle \dot{p}_l \rangle = \frac{dD_{QL}}{dp} - \frac{D_{QL}}{kI} \quad (33)
\]

Fokker-Planck equation

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial p} \left( D_{QL}(p) \frac{\partial f}{\partial p} \right) + \frac{\partial}{\partial p} \left( \frac{D_{QL}}{kI} f \right) \quad (34)
\]

Usual quasilinear diffusive term complemented with a friction term due to spontaneous, or Cherenkov, emission
Weak warm beam instability

• 1962 (1961) QL theory
• Perturbative theory +random phase approximation
• Landau growth of waves
• Diffusion of particles

\[ \partial_t f = \partial_v (D_{QL}(t, v) \partial_v f), \]
\[ \partial_t \psi = 2\gamma_L(t, v) \psi, \]

No dissipation!
Spontaneous emission triggers the walk toward thermal equilibrium

If spontaneous emission is taken into account, it forbids a plateau on the tail of the velocity distribution to be stationary, and triggers the walk toward the eventual thermal distribution.
Conclusion
We introduced a Hamiltonian description of the coupled system of $N$ tail particles and $M$ Langmuir waves.

The wave phase and amplitude evolutions were computed by perturbation theory in the coupling parameter of particle and wave dynamics.

Together with the collective Landau damping, the calculation derived also the spontaneous emission of waves by particles.

Landau damping turned out to be a relaxation mechanism driving waves to their thermal level.
When extending the perturbation calculation to a broad spectrum, the classical quasilinear equations are recovered with a correction due to spontaneous emission.

Spontaneous emission brings a contribution to the walk toward equilibrium.
References


Resonant particles may experience trapping or chaotic dynamics

\[ k \cdot \Delta r(t) \sim 2\pi \text{ or larger} \]

For such particles previous linearizations not appropriate

These linearizations still justified for non-resonant particles over times of trapping and chaos for resonant ones
Bulk and tail particles
Split the set of $N$ particles into bulk and tail in the spirit of Onishchenko et al. 1970 and O’Neil et al. 1971
Bulk: set of particles not resonant with Langmuir waves
Analysis leading to $N$-body linear equation for the electrostatic potential performed for the $N_{\text{bulk}}$ particle only
Keeping the exact contribution of the $N_{\text{tail}}$ particles
Tail particles from 1 to $N_{\text{tail}}$: set of integers $S_{\text{tail}}$

Bulk particles from $N_{\text{tail}} + 1$ to $N = N_{\text{bulk}} + N_{\text{tail}}$
Set of integers $S_{\text{bulk}}$

For $l \in S_{\text{bulk}}$

\[
\phi_l(m, t) = \phi_{\text{bulk}}^{(\text{bal})}(m, t) + \sum_{j \in S_{\text{bulk}}; j \neq l} \frac{ie}{\epsilon_0 k_m^2} \exp[-ik_m \cdot r_j^{(0)}(t)] k_m \cdot \Delta \xi_j + \frac{N_{\text{bulk}} - 1}{N_{\text{bulk}}} U(m, t),
\]

\[
U(m, t) = -\frac{eN_{\text{bulk}}}{\epsilon_0 k_m^2 (N_{\text{bulk}} - 1)} \sum_{j \in S_{\text{tail}}} \exp(-ik_m \cdot r_j),
\]
Previous calculation for the bulk particles only

\[ k_m^2 \phi(m, \omega) - \frac{\omega_p^2}{N_{\text{bulk}}} \sum_n k_m \cdot k_n \]

\[ \sum_{j \in S_{\text{bulk}}} \frac{\phi(n, \omega + \omega_{n,j} - \omega_{m,j})}{(\omega - \omega_{m,j})^2} \exp[i(k_n - k_m) \cdot r_{j0}] \]

\[ = k_m^2 \phi^{(\text{bal})}_{\text{bulk}}(m, \omega) + k_m^2 U(m, \omega). \] (4)

where \( \omega_{l,j} = k_l \cdot v_j \)
Introducing a smooth velocity distribution for the bulk

$$\epsilon_{\text{bulk}}(m, \omega) \Phi(m, \omega) = \phi_{\text{bal}}^{\text{bulk}}(m, \omega) + U(m, \omega) \quad (5)$$
Amplitude equations

For scales larger than $\lambda_D$, the electric potential for the bulk is a superposition of Langmuir waves.

Tail particles slightly modify these waves: deriving an amplitude equation for $\Phi(m, t)$ in a way similar to O’Neil et al. 1971 and Onishchenko et al. 1970.
Tail particles slightly modify these waves: deriving an amplitude equation for $\Phi(m, t)$ in a way similar to O’Neil et al. 1971 and Onishchenko et al. 1970.

Frequency for wavevector $k_m$ close to $\omega_m$ solving $\epsilon_{\text{bulk}}(m, \omega_m) = 0$

$\omega_m$ is real

$\Phi_{\text{bulk}}(m, \omega)$ solution for $U(m, \omega) = 0$

$$\epsilon_{\text{bulk}}(m, \omega)(\Phi(m, \omega) - \Phi_{\text{bulk}}(m, \omega)) = U(m, \omega). \quad (6)$$

$\Phi(m, t)$ close to $\Phi_{\text{bulk}}(m, t) = A \exp(-i\omega_m t)$

of the case $N_{\text{tail}} = 0$

$$A = -\sum_{j \in S_{\text{bulk}}} \frac{e}{\epsilon_0 k_m^2 \epsilon_m} \frac{\exp[-ik_m \cdot r_j(0)]}{\omega_m - k_m \cdot r_j(0)}$$
\[ \epsilon_{\text{bulk}}(m, \omega)(\Phi(m, \omega) - \Phi_{\text{bulk}}(m, \omega)) = U(m, \omega) \]

Define \( g(m, t) = \frac{\Phi(m, t)}{\Phi_{\text{bulk}}(m, t)} \) with \( g(m, 0) = 1 \)

Then \( \Phi(m, \omega) = A \ g(m, \omega - \omega_m) \)

\[ A \ \epsilon_{\text{bulk}}(m, \omega_m + \omega') \left[ g(m, \omega') - \frac{i}{\omega'} \right] = U(m, \omega_m + \omega'), \]

where \( \omega' = \omega - \omega_m \)

For \( N_{\text{tail}} \ll N_{\text{bulk}} \) \( g(m, t) \) slowly evolving complex amplitude
Dominant part of $g(m, \omega)$ concentrated near zero

$$A \epsilon_{\text{bulk}}(m, \omega_m + \omega') \left[ g(m, \omega') - \frac{i}{\omega'} \right] = U(m, \omega_m + \omega')$$

Taylor-expanding $\epsilon_{\text{bulk}}(m, \omega_m + \omega')$ about $\omega' = 0$

$$\frac{\partial \epsilon_{\text{bulk}}(m, \omega_m)}{\partial \omega} \omega'$$ to lowest order

Setting this into Eq. (7) and performing the inverse Laplace transform

Amplitude equation for $\Phi(m, t)$

$$\frac{\partial \Phi(m, t)}{\partial t} + i\omega_m \Phi(m, t) = \frac{ieN_{\text{bulk}}}{\epsilon_0 k_m^2 (N_{\text{bulk}} - 1) \frac{\partial \epsilon_{\text{bulk}}}{\partial \omega}(m, \omega_m)} \sum_{j \in S_{\text{tail}}} \exp(-ik_m \cdot r_j)$$

similar to O’Neil et al. 1971 and Onishchenko et al. 1970
\[
\frac{\partial \Phi(m, t)}{\partial t} + i\omega_m \Phi(m, t) \\
= \frac{ie N_{bulk}}{\epsilon_0 k_m^2 (N_{bulk} - 1)} \frac{\partial \epsilon_{bulk}}{\partial \omega}(m, \omega_m) \sum_{j \in S_{tail}} \exp(-ik_m \cdot r_j)
\]  

Self-consistent wave-particle dynamics

Self-consistent dynamics of \( M \) Langmuir waves and of the tail particles ruled by this equation written for each wave and by the equation of motion of these particles due to the \( M \) waves

\[ \ddot{r}_j = \text{summation over the indices n of the} \ M \text{waves} \]
Tail-tail interactions neglected owing to the low density of the tail particles.

Generalizes to 3D the self-consistent dynamics in Mynick and Kaufman 1978
in Tennyson, Meiss, Morrison 1994
in Antoni, Elskens and DFE 1998
in Elskens and DFE 2003
obtained without using any continuous approximation
direct mechanical reduction of degrees of freedom
starting with the $N$-body problem.