The inter-particle distance, an impact parameter no longer overlooked for the computation of Coulomb scattering

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Summary

• Introduction
  What do we call collisions and what is their effect?
• Mathematical model
• Perturbative approximation to weak shielded Coulomb deflections
• Rutherford description for close collisions
• Obvious matching of the descriptions of close and far deflections
Collisional transport described in textbooks by two opposite point of views

1. The **two-body Rutherford collision picture**
   Describes correctly collisions for impact parameters \( b \ll \text{interparticle distance} \, d \)
   Transport coefficients computed by an \textit{ad hoc extension} of the integrals over \( b \) up to Debye length \( \gg d \)
   Yields the Coulomb logarithm + uncertainty

2. A **mean-field approach based on the Balescu-Lenard equation**
   Describes correctly collisions for \( b \gg d \)
   Transport coefficients computed by an \textit{ad hoc extension} of the integrals over \( b \) down to the classical distance of minimum approach \( \ll d \)
   Yields the Coulomb logarithm + uncertainty
Historical path

Sixty years ago, two groups at UC Berkeley's Radiation Laboratory:
• simultaneously studied transport due to collisions in non-magnetized plasmas
• quoted each other's results in their respective papers

Gasiorowicz, Neuman and Riddell 1956 dealt with the mean-field part of the interaction using perturbation theory in electric field amplitude.

Rosenbluth, MacDonald and Judd 1957 used the Rutherford picture of two-body collisions.
Historical path

Within the same approximations as Gasiorowicz et al. 1956: a more elegant derivation using Balescu-Lenard equation for instance: Hazeltine and Waelbroeck 2004

Single calculation providing both dynamical friction and diffusion coefficient

Mean field description ➔ no two-body collisions

Smooth and progressive deflection due to many contributions Natural for $b \gg d$

“Collisional” transport involves:
- Mostly deflections due to the simultaneous and lasting action of most particles in the Debye sphere
- Secondarily two-body time localized collisions

“Collisional” transport: weakly collisional in reality

More like “short range induced transport”
The interparticle distance, an overlooked scale

Rosenbluth, MacDonald and Judd or Balescu-Lenard

RMJ  Interparticle distance  d  BL

Ad hoc cutoff in each approach
Transport coefficients depend only logarithmically on these cutoffs
Forgetting about the physically relevant ones
considering in both cases the scales between $\lambda_{ma}$ and $\lambda_D$
two results found to agree (Gasiorowicz et al. 1956, Rosenbluth et al. 1957)
Quoting each other
Brought confidence in these complementary extrapolations
Present description of collisional transport in plasmas in many plasma physics textbooks
Classical and new descriptions of collisional transport

Balescu-Lenard and Rosenbluth, MacDonald and Judd

RMJ  Interparticle distance  d  BL

No man’s land

New theory: N-body system ruled by Newton’s second law with shielded Coulombian potentials; no test particle

Perturbation theory in the electric field $E$

Approximate Rutherford deflection

Matching with two-body Rutherford deflection

Provides a full description for all b’s

Convergent one thanks to shielding (due to collisions!)

Coulomb logarithm +O(1)
Structure of the derivation

Deflection of particle computed perturbatively, except for close encounters. 3 steps:

1. First order perturbation theory in $E$
total deflection = sum of individual deflections due to all other particles
For $\lambda_{ma} << b << \lambda_D$, deflection due to a particle = Rutherford deflection due to this particle as if it were alone

2. For $b \sim \lambda_D$ deflection given by the Rutherford expression multiplied by some function of the impact parameter reflecting shielding (bare potential substituted with its “dressed" one)

3. In a close encounter, the deflection is exactly Rutherford deflection as if the other $N-2$ particles were absent

Conclusion: matching the 3 derivations yields a convergent expression, since deflections decay rapidly for $b > \lambda_D$
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Plasma in thermal equilibrium, uniform density
Random initial positions of particles
⇒ no collective aspect in the dynamics of particles
dynamics ruled by the cumulative effect of two-body deflections
(not necessarily collisions)
Random initial positions $r_{i0}$’s
Initial velocities $v_{i0}$, such that the smoothed velocity distribution averaged over a Debye sphere is close to a Maxwellian
$N$ electrons in a cube with size $L$
Periodic boundary conditions
(electrons in a uniform neutralizing background
$L \to \infty$, $N \to \infty$ with constant particle density
$n = N/L^3$
(hence constant Debye length $\lambda_D$)
Electrons with altered ballistic motion
$r_i(t) = r_{i0} + v_{i0}t + \delta r_i(t)$
Shielded electric field for one mechanical realization

Interaction described by shielded electric field

\[ \delta \ddot{r}_l = \sum_{j=1; j \neq l}^{N} a(r_l - r_j, v_j), \quad (1) \]

\[ a(r, v) = \frac{e}{m_e} \nabla \Phi(r, v), \quad (2) \]

\[ \Phi(r, v) = -\frac{e}{L^3 \epsilon_0} \sum_m \frac{\exp(i k_m \cdot r)}{k_m^2 \epsilon(m, k_m \cdot v)}. \quad (3) \]

\( \epsilon(k_m, \omega) \) dielectric function

Wave vectors \( k_m = 2m/L \)

Sum over all vectors \( m = (m_x, m_y, m_z) \) with three integer components
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$\delta \vec{r}_l$ computed by first order perturbation theory in $\Phi$

Ballistic motion $\vec{r}_l^{(0)}(t) = \vec{r}_{l0} + \vec{v}_{l0} t$ zeroth order approximation

$$\delta \vec{r}_l(t) = \sum_{j=1; j\neq l}^{N} \delta \vec{r}_{lj}(0, t),$$

(4)

$$\delta \vec{r}_{lj}(t_1, t_2) = \int_{t_1}^{t_2} a[\vec{r}_l^{(0)}(t') - \vec{r}_j^{(0)}(t'), v_j] \, dt'.$$

(5)

Relative ballistic motion

$$\vec{r}_l^{(0)}(t') - \vec{r}_j^{(0)}(t') = b_{lj} + \Delta v_{lj}(t' - t_{lj}),$$

(6)

$t_{lj}$ time of closest approach of the two ballistic orbits

Works already for $N=2$ : Rutherford scattering
\( \vec{b}_{lj} \) vector joining particle \( j \) to particle \( l \) at \( t_{lj} \)

\( b_{lj} = \| \vec{b}_{lj} \| \) impact parameter of orbits \( j \) and \( l \) when singled out

Random initial positions \( \Rightarrow \) random values for \( b_{lj} \) and \( t_{lj} \)
Trace of the diffusion tensor for the particle velocities computed for brevity 
Average over all the $r_{l0}$’s

$$
\langle \| \delta \vec{r}_l(t) \| ^2 \rangle = \sum_{j=1; j \neq l}^N \langle \| \delta \vec{r}_{lj}(t) \| ^2 \rangle,
$$

(7)

Initial positions and $r_i - r_j$’s for $i \neq j$ independently random No contribution of two different particles
Simultaneous scattering of particle $l$ with the many particles inside its Debye sphere
However $\langle \| \delta \vec{r}_l(t) \| ^2 \rangle = \text{sum of individual two-body deflections}$
for $b_{lj}$’s such that first order perturbation theory is correct
For $b_{lj} \ll \lambda_D$, main contribution of $a[\vec{r}_i^{(0)}(t') - \vec{r}_j^{(0)}(t'), v_j]$ to the deflection of particle $l$ comes from times $t'$ at which $||\vec{r}_i^{(0)}(t') - \vec{r}_j^{(0)}(t')|| \ll \lambda_D$

$\Rightarrow a(r,v)$ takes on its bare Coulombian value

The contribution to $\langle ||\delta \vec{r}_i(t)||^2 \rangle$ of particles with given $b_{lj}$ and $\Delta v_{lj}$ can be computed as if it would result from successive two-body collisions as is done in (Rosenbluth et al. 1957) and in many textbooks
Perturbation theory in the electric field $E$
Convergent one thanks to shielding (due to collisions!)
Approximate Rutherford deflection
Matching with two-body Rutherford deflection
Small deflections
For impact parameters of the order of $\lambda_D$
deflection due to particle $j$ to be computed
with shielded potential
For simplicity, calculation for the case where
$v_j$ is small
so that $\Phi(r, v) \simeq \Phi(r, 0)$, Yukawa potential
$\Phi_Y(r) = -e (4 \epsilon_0 \|r\|)^{-1} \exp(-\|r\|/\lambda_D)$
In the small deflection limit, standard calculation using the fact that the force derives from a central potential

⇒ full deflection of particle $l$ due to particle $j$

$$
\delta \mathbf{r}_{lj}(-\infty, +\infty) = \frac{e^2}{4m_e\varepsilon_0} b_{lj} \int_{-\infty}^{+\infty} \left[ \frac{1}{r^3(t)} + \frac{1}{\lambda_D r^2(t)} \right] \exp\left[-\frac{r(t)}{\lambda_D}\right] \, dt
$$

(8)

$$
r(t) = (b_{lj}^2 + \Delta \nu_{lj}^2 t^2)^{1/2}
$$

Let $\theta = \arcsin[\Delta \nu_{lj} t / r(t)]$

$$
\delta \mathbf{r}_{lj}(-\infty, +\infty) = -\frac{2e^2}{4m_e\varepsilon_0 \Delta \nu_{lj}} \frac{h(b_{lj})}{b_{lj}^2} b_{lj},
$$

(9)

$$
h(b) = \int_0^{\pi/2} \left[ \cos \theta + \frac{b}{\lambda_D} \right] \exp\left[-\frac{b}{\lambda_D \cos \theta}\right] \, d\theta
$$

(10)
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Case \( b_{lj} \sim \lambda_{ma} \)

We write the acceleration of particle \( l \) as

\[
\ddot{r}_l = \dot{a}(r_l - r_j, v_j) + \sum_{p=1; p \neq l, j}^{N} \dot{a}(r_l - r_p, v_p) \quad (11)
\]

\[
\ddot{r}_j = \dot{a}(r_j - r_l, v_l) + \sum_{p=1; p \neq l, j}^{N} \dot{a}(r_j - r_p, v_p) \quad (12)
\]

Distance between two particles \( \ll d = n^{-1/3} = N^{-1/3}L \Rightarrow \)

accelerations imparted to them by all other particles are almost equal
Subtracting the two equations of motion ⇒
two summations over $p$ almost cancel

$$\frac{d^2(r_l - r_j)}{dt^2} = 2a(r_l - r_j), \quad (13)$$

Close particles ⇒ shielded potential = bare Coulomb one

Equation describing Rutherford collision in
the centre-of-mass frame
in the absence of all other particles
Typical duration of the deflection

$b_{lj} \ll d, \Delta t_{lj} \ll \Delta t_{lp}$ for all $p$'s

$\Rightarrow N - 2$ other particles produce a negligible deflection

of the centre of mass during the Rutherford two-body collision

$\Rightarrow$ deflection of particle $l$ exactly that of a Rutherford two-body collision
Contribution of such collisions to $\langle \| \delta \vec{r}_i(t) \|^2 \rangle$ was calculated in (Rosenbluth et al. 1957)
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Deflection of particle $l$ due to particle $j$ as computed by the perturbation theory = approximation to the Rutherford deflection for the same $b$

⇒ perturbative deflection approximated with the full Rutherford one

⇒ obvious matching of the theories for $b_{lj} \sim \lambda_{ma}$ and for $\lambda_D \gg b_{lj} \gg \lambda_{ma}$

Estimate of (Rosenbluth et al. 1957) may be used in the whole domain $b_{lj} \ll \lambda_D$

$$\delta \mathbf{r}_{lj}(-\infty, +\infty) = -\frac{2e^2}{4\pi m_e \epsilon_0 \Delta v_{lj}} \frac{h(b_{lj})}{b_{lj}^2} b_{lj},$$

Corrected for larger impact parameters For $\lambda_D$ infinite $h_0(b) = 1$

$$h(b) = \int_0^{\pi/2} \left[ \cos \theta + \frac{b}{\lambda_D} \right] \exp\left[-\frac{b}{\lambda_D \cos \theta}\right] d\theta$$
Diffusion of velocities

\[ \langle \| \delta \mathbf{r}_l \|^2 \rangle_{\Delta v_{lj}} = \left( \frac{e^2}{2\pi m_e \epsilon_0 \Delta v_{lj}} \right)^2 2\pi \Delta v_{lj} t I(\lambda_{ma}) \]

Rosenbluth et al. (1957) estimate

\[ I_0(\lambda_{ma}) = \int_{\lambda_{ma}}^{\infty} h_0^2(b)/b \, db = \int_{\lambda_{ma}}^{\lambda_D} 1/b \, db = \ln \frac{\lambda_D}{\lambda_{ma}} \]

For \( \lambda_D \) infinite \( h_0(b) = 1 \)

The real value of the logarithm differs by a number of order 1: generally no impact
Calculation of dynamical friction requires second order perturbation theory follows the same lines as those for the diffusion coefficient
Conclusion

Perturbation theory in the electric field $E$
Convergent one thanks to shielding (due to collisions!)
Approximate Rutherford deflection
Matching with two-body Rutherford deflection
First description of “collisional” transport incorporating all impact parameters without any ad hoc cut-off with clear physical justification

Good news: no dramatic change in the final result

2/3 of collisional transport is due to scales less than $d$

1/3 of collisional transport is due to scales larger than $d$

“Collisional” transport: rather “short range induced transport”, “transport due to unshielded Coulomb interactions”

Too long!: keep the name, change the view

Debye shielded potential due to Coulomb deflections

Yields a description of pair interaction which provides a direct calculation of short range induced (“collisional”) transport: Shielding and collisions are intimately linked
Part of the global deflection due to a single particle
The global deflection is due to many particles:
- diffusion of velocities
- no change in density
Each particle interacts simultaneously with many other ones on the Debye length scale. Suggests the need for a collective description. However, transport effect of these interactions well approximated by a sum of independent binary estimates because the deflections are so weak that they can be treated perturbatively at lowest order.

There is a smooth connection between impact parameters where the two-body Rutherford picture is correct, and those where a collective description is mandatory.
New theory: N-body system ruled by Newton’s second law with shielded Coulombian potentials; no test particle (2014)

Shielded Coulombian potential computed till then by a kinetic approach

N-body calculation based on a kinetic result!

Incentive to tackle anew the dynamics of N electrons in Coulombian interaction by following a path analogous to Gasiorowicz, Neuman and Riddell, 1956

Direct path from N-body dynamics uncovered

Heuristic path: creative ignorance*: Serendipidity
Open issues

• Diffusion is computed, but chaos is overlooked: origin of decorrelation to be exhibited
• ???

知之为知之，不知为不知，是知也。
(zhī zhī wèi zhī zhī, bù zhī wèi bù zhī, shì zhī yě)
“Real knowledge is to know the extent of one’s ignorance.”
- Confucius

writtenchinese.com
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